

MINIMAL POLYNOMIALS FOR THE COORDINATES OF THE HARBORTH GRAPH

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ABSTRACT. The Harborth graph is the smallest known example of a 4-regular planar unit-distance graph. In this paper we give an analytical description of the coordinates of its vertices for a particular embedding in the Euclidean plane. More precisely, we show how to calculate the minimal polynomials of the coordinates of its vertices with the help of a computer algebra system, and list these. Furthermore some algebraic properties of these polynomials, and consequences to the structure of the Harborth graph are determined.

1. INTRODUCTION

The Harborth graph (see Figure 1) is the smallest known example of a 4-regular planar unit-distance graph. That is a planar graph, all of which edges are of unit length, with exactly four edges meeting in each vertex. This graph was named after its discoverer H. Harborth, who first presented it to the general public as a research problem in [4] and to a large international audience in a talk at the Eugène Strens Memorial Conference on Recreational Mathematics and its History in 1986 [5, 12]. At both occasions he posed the question, if a smaller example of a 4-regular planar unit-distance graph could be found.

Curiously enough, up until now nearly all published pictures of the Harborth graph – even the original ones in [4, 5], as well as those in textbooks [6, 7] – seem to be slightly vague and inaccurate, with the vertices always being depicted by large dots. Furthermore there has not been given any analytical description of the Harborth graph, yet. That is, if we consider the graph as embedded in the Euclidean plane with a given coordinate system, the coordinates of the vertices have never been calculated exactly. This has gone to the point that the Harborth graph was even thought by some to be nonrigid¹, which, as the results of this paper imply, cannot be the case. Thus the wish for an “exact” description remained. This wish has recently been expressed again in the world wide web [11], which prompted this paper.

With the advent of dynamical geometry systems, several authors were finally able to produce more precise pictures [14, 11], which led to further evidence that there is one unique realisation of the Harborth graph in the Euclidean plane. With this paper we go one step further: using one particular way of construction, we set up a set of quadratic equations which completely describe the coordinates of

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¹ If we consider the Harborth graph as a (mechanical) framework consisting of rigid bars interconnected by rotatable joints, nonrigidity means that some vertices can be moved with respect to each other, that is the whole framework allows motions which are different from congruences [13].

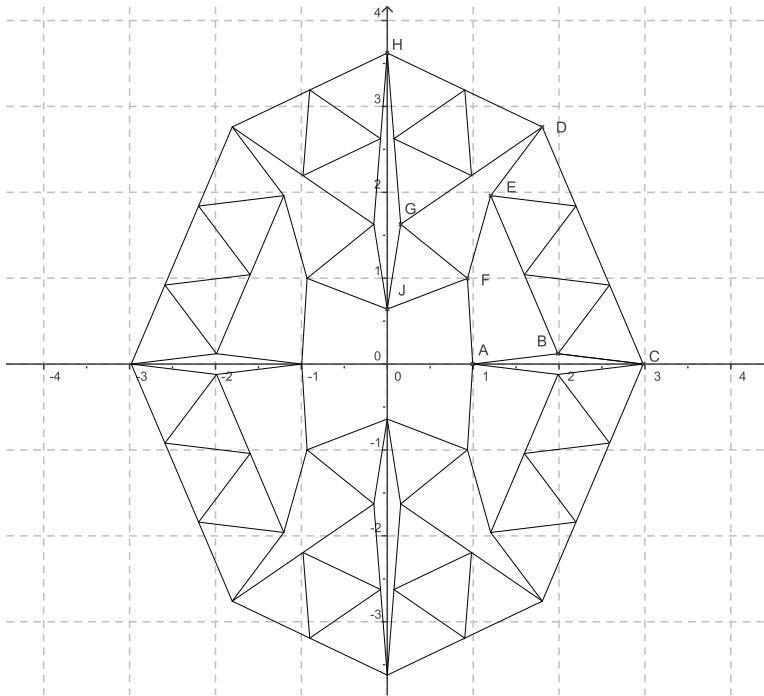


FIGURE 1. The Harborth graph embedded in the Euclidean plane.

crucial vertices of the Harborth graph. Using this initial set of equations, we will show that all the coordinates are algebraic numbers, and we will calculate their minimal polynomials with the help of a computer algebra system. Even though, as we will see later on, it is impossible to solve the corresponding algebraic equations exactly, which in our understanding means in terms of radicals, nevertheless we thus have achieved an exact analytic description of the Harborth graph, since, together with easy to calculate numerical approximations for the actual coordinates, these polynomials uniquely determine each coordinate.

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2. USING DYNAMIC GEOMETRY SOFTWARE AND NUMERICAL ANALYSIS

2.1. Geometric construction. Because of the obvious twofold symmetry of the Harborth graph, it is enough to analyse one of its quarters. Therefore, in a first step we construct one of these quarters (see Figure 2), using one of the existing (imprecise) first generation pictures as a blueprint.

We start from an initial isosceles triangle ABC of fixed but arbitrary height T , with two sides being of unit length, and a neighbouring symmetric trapezoid $BCDE$, which has the side BC with the initial triangle in common. The parallel sides of the trapezoid are chosen to be of length 2 and 3 respectively. The remaining

points are constructed from this initial configuration by using compass and ruler techniques. In the following, we list the necessary steps. Thereby $\text{Circ}(P, r)$ denotes the circle with center P and radius r , and \cap the operation of letting two geometric figures intersect. Thus we get

$$(2.1) \quad F := \text{Circ}(A, 1) \cap \text{Circ}(E, 1)$$

$$(2.2) \quad G := \text{Circ}(F, 1) \cap \text{Circ}(D, 2)$$

$$(2.3) \quad H := \text{Circ}(D, 2) \cap \text{Circ}(G, 2)$$

$$(2.4) \quad J := \text{Circ}(F, 1) \cap \text{Circ}(G, 1)$$

Although the intersection of two circles usually consists of two points, we use this notation as if there was no ambiguity. We are allowed to this, because we choose the resulting points of intersection according to our blueprint. Thus, e.g., F is chosen in such a way that the quadrangle $ABEF$ is convex. In the sequel we will call the configuration of the points A to J thus constructed the **Harborth configuration**. We will do this, even if the parameter T initially has not been chosen correctly, so that the configuration cannot be completed to the whole Harborth graph.

Clearly, proceeding as described, with using an arbitrary nonnegative value for the height T of the initial triangle, the line through the final crucial points H and J will only by (a very small) chance meet the line through A and C at an angle φ of 90° . But, in order that the Harborth configuration can be completed to the whole Harborth graph, we have to make sure that this last condition is satisfied.

Using the dynamic geometry software GeoGebra [8], we are able to manipulate the Harborth configuration in dependence on the parameter T . Furthermore with GeoGebra we are able to read off approximate values (up to five decimal places) for the resulting angles and coordinates. At this early stage of our investigations already, some important observations can be made:

Remark 2.1.

- (1) The point F , and thus the whole Harborth configuration only exists for $T \in [0, b]$, where b is approximately 0.13504.
- (2) For $T \in [0, b]$, the angle φ between AC and HJ lies approximately in the interval $[85.88496^\circ, 94.59043^\circ]$.
- (3) For $T \simeq 0.12073$, the angle φ approximates 90° .

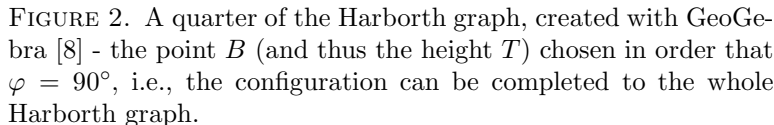
The upper bound b in the above remark is determined by the fact that for $T > b$ the distance between A and E becomes greater than 2, and the circles with centers A and E of radius 1 do not meet anymore.

As hinted at in the introduction, once we have calculated the exact values for the extremal angles - which we will do in one of the following sections - the observations collected in Remark 2.1 allow us to give the first analytic proof² of the planarity of the Harborth graph which does not resort to pictures or models only.

Theorem 2.2. *There exists $T \in [0, b]$ such that $\varphi = 90^\circ$ exactly, and the Harborth configuration can be completed to the Harborth graph.*

Proof. [10] Since the geometric operations we used in our construction (i.e., drawing circles and letting them intersect) depend continuously on their parameters (i.e.,

²The basic idea of this proof was first communicated to the author by H. Löwe in 2003. Here it is presented with his kind permission.



Corollary 2.3. *The Harborth graph indeed is a 4-regular unit-distance graph which is planar.*

2.2. Setting up algebraic equations. To describe the points of the Harborth graph more precisely we need to introduce coordinates. For the time being, we choose the point A as the center of our coordinate system, and the ray which extends the base side AC of the triangle ABC as the positive x -axis. When considering the Harborth graph as a whole, this is by far not the most obvious choice. In fact, in a later section we will use its center of symmetry as the origin of a more natural coordinate system, and then will have to "translate" our intermediate results. As

we proceed, we will see that in some places this will prove to be quite cumbersome, which will retroactively justify our initial choice of coordinates.

Now let (t, T) be the coordinates of the point B . Since A , B , and C form an isosceles triangle, the coordinates of the point C are given by $(2t, 0)$.

Denoting the coordinates of any point P of the graph different from B with (x_P, y_P) , we have:

$$(2.5) \quad t^2 + T^2 - 1 = 0$$

$$(2.6) \quad -t \cdot (x_D - 2 \cdot t) + T \cdot y_D - \frac{3}{2} = 0$$

$$(2.7) \quad (x_D - 2 \cdot t)^2 + y_D^2 - 9 = 0$$

$$(2.8) \quad t \cdot (x_E - t) - T \cdot (y_E - T) + 1 = 0$$

$$(2.9) \quad (x_E - t)^2 + (y_E - T)^2 - 4 = 0$$

$$(2.10) \quad x_F^2 + y_F^2 - 1 = 0$$

$$(2.11) \quad (x_E - x_F)^2 + (y_E - y_F)^2 - 1 = 0$$

$$(2.12) \quad (x_F - x_G)^2 + (y_F - y_G)^2 - 1 = 0$$

$$(2.13) \quad (x_D - x_G)^2 + (y_D - y_G)^2 - 4 = 0$$

$$(2.14) \quad (x_D - x_H)^2 + (y_D - y_H)^2 - 4 = 0$$

$$(2.15) \quad (x_G - x_H)^2 + (y_G - y_H)^2 - 4 = 0$$

$$(2.16) \quad (x_F - x_J)^2 + (y_F - y_J)^2 - 1 = 0$$

$$(2.17) \quad (x_G - x_J)^2 + (y_G - y_J)^2 - 1 = 0$$

$$(2.18) \quad x_H - x_J = 0$$

Let us shortly comment on the meaning of these equations: (2.6), (2.7), respectively (2.8), (2.9), define the vertices D and E of the trapezoid which do not belong to the initial triangle ABC . Equations (2.6) and (2.8) stem from the fact that the line BC meets CD at an angle of 60° and the line EB meets BC at 120° . Every further pair of Equations (2.10)-(2.17) is chosen in accordance to the geometric constructions described in (2.1) - (2.4), each pair defining one of the points F - J . Finally, Equation (2.18) has to be satisfied in order that the lines given by AC and HJ meet at an angle of 90° .

Using this set of equations, and approximations for the coordinates, which we read off from Figure 2 with the help of GeoGebra, we are already able to calculate arbitrary precise approximations of all coordinates by using standard numerical algorithms. E.g., the results below show the coordinates with an exactness of 15 digits; they were calculated with Mathematica, version 4.0.1.0.

$$(2.19) \quad \begin{aligned} B = (t, T) &\simeq (0.992685948824186, 0.120725337054926) \\ D &\simeq (0.809996600722107, 2.760161754567202) \\ E &\simeq (0.209102417540010, 1.960833173433061) \\ F &\simeq (-0.061398137844065, 0.998113354619244) \\ G &\simeq (-0.838419516770942, 1.627587561152422) \\ H &\simeq (-0.995049481192288, 3.621444891616507) \\ J &\simeq (-0.995049481192288, 0.639930204451542) \end{aligned}$$

3. DEDUCING AN EQUATION FOR THE y -COORDINATE OF THE POINT B

In this section we will deduce the minimal polynomial for the coordinate $y_B = T$, i.e., the unique primitive integer polynomial P_T of smallest degree such that $P_T(T) = 0$ holds [2, Section 4.1.1].

Let us shortly describe the main approach which we will repeatedly take: Given two polynomial equations, which are satisfied by certain coordinates, we will calculate the resultant of the corresponding polynomials with respect to one of the appearing variables, thus eliminating this particular variable. Sometimes, if the polynomials are not too complicated, we will use Groebner basis techniques to treat more than two polynomial equations simultaneously. Both procedures lead to new polynomials or polynomial equations, which are consequences of the original ones³, but contain fewer variables. To keep expressions from becoming too complicated, and running times from becoming too long we will try to factor the resulting polynomials. Many times it will prove to be advantageous to allow the factorization to be done over the ring extension $\mathbb{Z}[\sqrt{3}]$. If a particular polynomial is reducible we will continue our deliberations with that factor, which corresponds to the actual values of the coordinates. To check this, we use numerical approximations analogous to those given in Section 2.2, but which are precise to an error of $\epsilon = 10^{-100}$. Most calculations, especially those of resultants, factorizations, and numerical evaluations, were done with Mathematica, version 4.0.1.0.

The succession of eliminations will be determined by the order in which the corresponding points were constructed. E.g., in the initial isosceles triangle ABC due to the choice of the coordinate system all point coordinates are directly expressible in terms of T . Next we will determine polynomials which describe the connection between the coordinates of D and E , respectively, and the parameter T . After that the polynomials for the coordinates of F are calculated by using those for the coordinates of D and E and eliminating the variables in between. In principle, continuing this procedure would lead to polynomials in x_H and T , respectively x_J and T . Using the final equation $x_H = x_J$, one should be able to deduce one polynomial in the variable T alone. Unfortunately, due to the increasing complexity of expressions we were not able to continue this line of thought to its conclusion, but had to resort to an alternative way. Nevertheless we will try to push as far as possible with this approach, and come up with an alternative, when it proves to be necessary.

In the sequel, we will switch between listing the polynomials and the corresponding polynomial equations at will. When only a polynomial P is given it should be understood that the coordinates appearing in P satisfy the corresponding polynomial equation $P = 0$.

3.1. From A to F . Using the procedure `GroebnerBasis` with Equations (2.5) - (2.7) as input, like

```
GroebnerBasis[{t^2+T^2-1,-t*(xD-2*t)+T*yD-3/2,(xD-2*t)^2+yD^2-9},
               {t,xD,yD,T}]
```

one of the polynomials we get is

$$(3.1) \quad P_{y_D,T} := 27 - 36T^2 + 12T \cdot y_D - 4y_D^2.$$

³For a more precise description see, e.g., [3] or [9].

Analogously, by changing the order of variables,

$$0 = 1 - 56T^2 + 784T^4 - 8x_D^2 - 208T^2 \cdot x_D^2 + 16x_D^4$$

can be deduced. This last equation is irreducible over \mathbb{Z} , but factors over $\mathbb{Z}[\sqrt{3}]$ into polynomials, which are quadratic in x_D :

$$(3.2) \quad (-1 + 28T^2 - 12\sqrt{3}Tx_D + 4x_D^2) \cdot (-1 + 28T^2 + 12\sqrt{3}Tx_D + 4x_D^2).$$

Using numerical results for T and x_D in analogy to (2.19), we see that only the first of these polynomials

$$(3.3) \quad P_{x_D, T} := -1 + 28T^2 - 12\sqrt{3}Tx_D + 4x_D^2$$

leads to the correct result. Solving (3.3) and (3.1) for x_D and y_D , respectively, and again discarding those solutions which do not describe the correct coordinates, we get explicit descriptions for the coordinates of D in terms of the parameter T :

$$(3.4) \quad x_D = \frac{1}{2} \left(3\sqrt{3}T + \sqrt{1 - T^2} \right),$$

$$(3.5) \quad y_D = \frac{3}{2} \left(T + \sqrt{3}\sqrt{1 - T^2} \right).$$

Starting with Equations (2.5), (2.8) and (2.9) and proceeding in the same manner as above, we are led to those polynomials which describe the dependence of the coordinates of the point E on the parameter T :

$$(3.6) \quad 3T^2 - x_E^2$$

and

$$(3.7) \quad P_{y_E, T} := -3 + 7T^2 - 4Ty_E + y_E^2.$$

Again, the corresponding equations can be explicitly solved:

$$(3.8) \quad x_E = \sqrt{3}T,$$

$$(3.9) \quad y_E = 2T + \sqrt{3}\sqrt{1 - T^2}.$$

Next we continue by calculating the coordinates for the point F , once more using Mathematica's **GroebnerBasis** function. This time we start with the newly found set of Equations (3.6) and (3.7), together with the Equations (2.10), (2.11) defining F . From this we get the following polynomials which describe the dependence of the coordinates x_F and y_F of F on the parameter T :

$$(3.10) \quad \begin{aligned} & -81 + 10800T^2 - 422496T^4 + 4272384T^6 - 19194112T^8 + 45801472T^{10} - \\ & 63111168T^{12} + 48234496T^{14} - 16777216T^{16} + \\ & (1296 - 92448T^2 + 1645056T^4 - 9573888T^6 + 30072832T^8 - 57655296T^{10} + \\ & 66060288T^{12} - 29360128T^{14}) x_F^2 + (-7776 + 228096T^2 - 1555200T^4 + \\ & 5271552T^6 - 12189696T^8 + 15728640T^{10} - 9437184T^{12}) x_F^4 + \\ & (20736 - 152064T^2 + 331776T^4 - 196608T^6 - 1310720T^8 + 2097152T^{10}) x_F^6 + \\ & (-20736 + 110592T^2 - 442368T^4 + 786432T^6 - 1048576T^8) x_F^8, \end{aligned}$$

$$(3.11) \quad \begin{aligned} P_{y_F, T} := & 81 - 648T^2 + 144T^4 - 2304T^6 + 4096T^8 + \\ & (432T - 864T^3 + 6528T^5 - 10240T^7) y_F + \\ & (-216 + 1584T^2 - 5376T^4 + 9216T^6) y_F^2 + \\ & (-576T + 1536T^3 - 4096T^5) y_F^3 + (144 - 384T^2 + 1024T^4) y_F^4. \end{aligned}$$

Once again, the first of these polynomials factors over $\mathbb{Z}[\sqrt{3}]$ into two polynomials of total degree 8 and degree 4 in x_F . Using the numerical values for T and x_F , we

can deduce that only one of these describes the connection between the variables T and x_F . It is

$$(3.12) \quad \begin{aligned} P_{x_F, T} := & -9 + 600T^2 - 3472T^4 + 5888T^6 - 4096T^8 - \\ & 8\sqrt{3} (9T + 96T^3 - 112T^5 + 256T^7) x_F + 8 (9 - 150T^2 + 256T^4 - 640T^6) x_F^2 + \\ & 8\sqrt{3} (36T - 96T^3 + 256T^5) x_F^3 + 16 (-9 + 24T^2 - 64T^4) x_F^4. \end{aligned}$$

As said before, trying to continue like this to calculate polynomials for the remaining points G, H and J , will lead into a deadend, because the resulting equations become too unwieldy to handle, and take too much time to calculate, even with the help of Mathematica. Still, our main goal remains to find one single equation describing the parameter T alone. Consequently we have to take a step back, and use a slightly more indirect approach, which we will describe in the section following the next one.

3.2. Interlude: Calculating the extremal values for which the Harborth configuration exists. With Equations (3.8) and (3.9) thus available, we are able to calculate the exact maximal value for T , hinted at in Remark 2.1. To this end, we first observe that in case of T being maximal the line segments AF and FE together, again form a straight line segment of twice the original length. Therefore, for T maximal, the coordinates of the point E satisfy $x_E^2 + y_E^2 - 4 = 0$. This, together with (3.8) and (3.9), after some small calculation leads to

$$(3.13) \quad 64T^4 - 56T^2 + 1 = 0.$$

Solving for T , and comparing with the numerical values presented in Remark 2.1, gives

Lemma 3.1. *The minimal and maximal value for T such that the Harborth configuration exists are $T = 0$, and*

$$(3.14) \quad T = \frac{1}{4} \sqrt{7 - 3\sqrt{5}},$$

respectively.

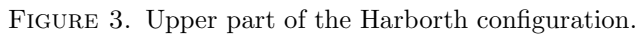
Using basic trigonometry, from this we are able to further deduce exact values for the crucial angle φ for extremal T :

In case of $T = 0$, the points A, B , and C lie on one line. So do the points D, E , and F . Moreover the intersection point Z of these two lines together with C and D form an equilateral triangle, the sides of which have length 3. Angle β (see Figure 2) becomes one of the angles of this triangle, and thus is equal to 60° . Furthermore the points D, F , and G form an isosceles triangle, with the length of the base side FG being one, and the other length being two. An analysis of the triangle formed by the lines DH , and the prolongations of DF and HJ , which contains the triangle DFG completely, allows us to calculate α . Since $\varphi = \alpha + \beta$ some further calculations show

Corollary 3.2. *For $T = 0$, the angle φ in the Harborth configuration is the unique solution of*

$$(3.15) \quad \sin(\varphi) = \frac{1}{4} (7 + 3\sqrt{5}) \sqrt{\frac{3}{22 + 6\sqrt{5}}}$$

in the interval $[0, 90^\circ]$, which up to an error of 10^{-15} is 85.884964999269942° .



Corollary 3.3. *For $T = \frac{1}{4}\sqrt{7-3\sqrt{5}}$, the angles α and β in the Harborth configuration (see Figure 2) are the unique solutions of*

and

in the interval $[0, 90^\circ]$.

3.3. From the points D and F to the points H and J . Now we continue with our task of determining the minimal polynomial for that particular T , for which the Harborth graph exists, i.e., for which $\varphi = 90^\circ$ holds. Our trick is, not to calculate the coordinates of the points H and J directly in dependence of the second coordinate T of the point B , but to introduce further variables X and Y which will lead to simpler equations. These new variables themselves will depend on the points D and F .

For this, let us consider F as the origin of a new coordinate system, and the line FD as the new x -axis (see Figure 3). Let (s, S) be the coordinates of the point E with respect to this new coordinate system. Clearly, since DEF forms an isosceles triangle, D is described by the coordinates $(2s, 0)$. Proceeding as above by using

Equations (2.12)-(2.17) in this new context, we successively get⁴:

$$(3.16) \quad 0 = 3 - 4s^2 + 4sx_G,$$

$$(3.17) \quad 0 = 9 - 40s^2 + 16s^4 + 16s^2y_G^2,$$

$$(3.18) \quad 0 = 9 - 48s^2 + 48s^4 + (12s - 48s^3)x_H + 16s^2x_H^2,$$

$$(3.19) \quad 0 = -81 - 144s^2 - 352s^4 - 256s^6 - 256s^8 + (144s^2 + 896s^4 + 256s^6)y_H^2 - 256s^4y_H^4,$$

$$(3.20) \quad 0 = 9 - 36s^2 + 16s^4 + (12s - 16s^3)x_J + 16s^2x_J^2,$$

$$(3.21) \quad 0 = 81 - 504s^2 + 1072s^4 - 896s^6 + 256s^8 + (-144s^2 + 256s^4 - 256s^6)y_J^2 + 256s^4y_J^4.$$

All these equations can be easily solved for the respective coordinates. In each case only one of the solutions is in accordance with our geometric construction. Below we present explicit formulas only for the coordinates of the points H and J :

$$(3.22) \quad x_H = \frac{-3 + 12s^2 + \sqrt{3}\sqrt{-9 + 40s^2 - 16s^4}}{8s},$$

$$(3.23) \quad y_H = \frac{3\sqrt{3} + 4\sqrt{3}s^2 + \sqrt{-9 + 40s^2 - 16s^4}}{8s},$$

$$(3.24) \quad x_J = \frac{-3 + 4s^2 - \sqrt{3}\sqrt{-9 + 40s^2 - 16s^4}}{8s},$$

$$(3.25) \quad y_J = \frac{-3\sqrt{3} + 4\sqrt{3}s^2 + \sqrt{-9 + 40s^2 - 16s^4}}{8s}.$$

Therefore the slope of the line HJ with regard to DF as x -axis is

$$(3.26) \quad m_\alpha := \frac{y_J - y_H}{x_J - x_H} = \frac{3\sqrt{3}}{4s^2 + \sqrt{3}\sqrt{-9 + 40s^2 - 16s^4}}.$$

Now, again we consider the whole Harborth configuration: let new variables X and Y be defined by $X := x_D - x_F$ and $Y := y_D - y_F$, where now x_D , y_D , and x_F , y_F are interpreted as the coordinates of the points D and F with regard to the initial coordinate system. Then the squared length of the line segment DF is given by $X^2 + Y^2$. It follows that

$$(3.27) \quad 4s^2 = X^2 + Y^2,$$

and Equation (3.26) becomes

$$(3.28) \quad m_\alpha = \frac{3\sqrt{3}}{X^2 + Y^2 + \sqrt{-9 + 10(X^2 + Y^2) - (X^2 + Y^2)^2}}.$$

In order to be able to complete the Harborth configuration to the whole Harborth graph, the angle φ between the lines HJ and AC must be a right angle. On the other hand, we have $\varphi = \alpha + \beta$, where α is the angle between HJ and DF , and β denotes the angle between DF and AC , as shown in Figure 2. Thus the equality $\alpha = 90^\circ - \beta$ must hold. Since $0^\circ < \alpha, \beta < 90^\circ$, we have $\tan \alpha = \tan(90^\circ - \beta) = \frac{1}{\tan(\beta)}$. Thus the

⁴We advise the reader to keep in mind that, although we use the same notation as in Section 2.2, now the coordinates have to be interpreted within the new coordinate frame.

respective slopes satisfy $m_\alpha = 1/m_\beta$. The slope m_β of β is given within the first coordinate system by $m_\beta = Y/X$. This, together with (3.28), implies

$$(3.29) \quad \frac{3\sqrt{3}}{X^2 + Y^2 + \sqrt{-9 + 10(X^2 + Y^2) - (X^2 + Y^2)^2}} = \frac{X}{Y},$$

which some calculations show to be equivalent to

$$(3.30) \quad 0 = (X^2 + Y^2) \left(-27 + 30X^2 - 4X^4 + 6\sqrt{3}XY - 4X^2Y^2 \right).$$

This implies

$$(3.31) \quad 0 = 27 - 30X^2 + 4X^4 - 6\sqrt{3}XY + 4X^2Y^2.$$

We set $F(X, Y) := 27 - 30X^2 + 4X^4 - 6\sqrt{3}XY + 4X^2Y^2$.

Next we produce polynomials $P_{X,T}$ and $P_{Y,T}$ which describe the connection between T and the new parameters X , and Y , respectively. To do that, this time we use Mathematica's resultant and factorization facilities, as described above. Starting with the polynomial $X - x_D + x_F$, and the polynomials $P_{x_D,T}$ and $P_{x_F,T}$ given by (3.3) and (3.12), we are thus able to successively eliminate the variables x_D and x_F , and get

$$(3.32) \quad \begin{aligned} P_{X,T} := & 108T^2 - 684T^4 + 1344T^6 - 1344T^8 + \sqrt{3}(-36T + 300T^3 - 616T^5 + 1024T^7)X \\ & + (9 - 213T^2 + 496T^4 - 1216T^6)X^2 + \sqrt{3}(36T - 96T^3 + 256T^5)X^3 \\ & + (-9 + 24T^2 - 64T^4)X^4 \in \mathbb{Z}[\sqrt{3}][X, T]. \end{aligned}$$

In an analogous manner we deduce a polynomial $P_{Y,T}$ in $\mathbb{Z}[Y, T]$, which for the Harborth configuration describes the connection between these two variables:

$$(3.33) \quad \begin{aligned} P_{Y,T} := & 81 - 405T^2 + 900T^4 - 1008T^6 + 448T^8 + (54T - 54T^3 + 456T^5 - 512T^7)Y + \\ & (-54 + 207T^2 - 624T^4 + 576T^6)Y^2 + (-18T + 48T^3 - 128T^5)Y^3 + \\ & (9 - 24T^2 + 64T^4)Y^4. \end{aligned}$$

Finally we repeat this procedure with the polynomials $F(X, Y)$, $P_{X,T}$, and $P_{Y,T}$ to eliminate variables X and T . I.e., first we let Mathematica calculate the resultant of F and $P_{X,T}$ with regard to X . We will not present the result here; let it be enough to state that the result is a polynomial of degree 8 in Y , degree 32 in T and total degree 32, which, but for a constant factor, cannot be factored further by Mathematica, even when considered over $\mathbb{Z}[\sqrt{3}]$. With the help of Mathematica, we are able to calculate the resultant of this polynomial and $P_{Y,T}$ with respect to Y . This leaves us with a polynomial in the single variable T of order 156. Strangely enough, this final polynomial is reducible over $\mathbb{Z}[\sqrt{3}]$, its factors (up to an integer constant) being $(2T + \sqrt{3})$, $(2T - \sqrt{3})$, $(64T^4 - 24T^2 + 9)^6$, and three other integer polynomials of degree 22, 28 and 80, respectively. Here, we need only list the polynomial of degree 22, since this is the one which has the y -coordinate T of the point B of the Harborth graph as one of its real roots⁵:

⁵In fact it is the positive root of smallest modulus.

Theorem 3.4. *The minimal polynomial for the y -coordinate T of the vertex B of the Harborth graph is*

$$P_T := -492075 + 52356780T^2 - 1441635408T^4 + 12222052416T^6 - 60567699456T^8 + 189747007488T^{10} - 417660420096T^{12} + 607025037312T^{14} - 655053815808T^{16} + 446118756352T^{18} - 422064422912T^{20} + 437348466688T^{22}.$$

Since this polynomial is primitive, and irreducible over \mathbb{Z} , it is the minimal polynomial of $y_B = T$. Thus we have achieved our desired first main result. The rest of this paper is concerned with the determination of the minimal polynomials of the other coordinates, and some of their properties.

4. MINIMAL POLYNOMIALS FOR THE y -COORDINATES

4.1. The points D, E , and F . The method to calculate the minimal polynomial for each of the y -coordinates of the points D, E, F is very similar to what we have done in the last subsection. Let P denote one of the points D, E , or F . Using Formulas (3.1), (3.7), and (3.11), respectively, which describe the connection between the y -coordinate y_P and the parameter T by way of irreducible polynomials $P_{y_P, T} \in \mathbb{Z}[y_P, T]$ of absolute degree 2, 2 and 4, we let Mathematica calculate the resultant of $P_{y_P, T}$ and P_T from above with respect to T . The result is a polynomial in $\mathbb{Z}[y_P]$. It is of degree 44 in case of y_D and y_E and of degree 88 for y_F . The first two of these polynomials each split over \mathbb{Z} into two irreducible factors of degree 22. The polynomial for y_F splits into one component of degree 44 and – again – into two polynomials of degree 22. In each case one of these factors corresponds to the actual y -coordinate.

Theorem 4.1. *The minimal polynomials for y_D, y_E and y_F are*

$$P_{y_D} := -2470693585135788 + 1679453964496051893y_D^2 - 2462573171102886288y_D^4 + 1847147913929328048y_D^6 - 888334179987132288y_D^8 + 302241307009227264y_D^{10} - 74768143621533696y_D^{12} + 13516084620361728y_D^{14} - 1721332250836992y_D^{16} + 139442448236544y_D^{18} - 6126808596480y_D^{20} + 109337116672y_D^{22},$$

$$P_{y_E} := -387038865725307 + 255845547796716y_E^2 - 1080696123714384y_E^4 + 985178573370432y_E^6 + 290816529555456y_E^8 + 1229422640467968y_E^{10} - 399291497201664y_E^{12} - 226953868935168y_E^{14} - 145914316455936y_E^{16} + 84049703993344y_E^{18} - 9462031056896y_E^{20} + 437348466688y_E^{22},$$

$$P_{y_F} := -6156736033068 + 4132620043369020y_F^2 - 28069535202466347y_F^4 + 54174190167055116y_F^6 - 44321252355544320y_F^8 + 16893977313239424y_F^{10} - 3430375146685440y_F^{12} + 781964817629184y_F^{14} - 165954075623424y_F^{16} + 16400930701312y_F^{18} - 579898179584y_F^{20} + 27334279168y_F^{22}.$$

4.2. The points G, H , and J . When determining the minimal polynomials of the y -coordinates of the points G, H , and J , we have to keep in mind that in Section 3.3 we used a different coordinate system for determining their coordinates than for those of the points A to F . Thus first we will have to transform the former into y -coordinates within our original system, which was centered in A . Since in this section it is paramount not to confuse these systems, in the sequel we will denote coordinates with regard to the coordinate system centered in F with capital letters,

and those with regard to the original one centered in A with small letters. Thus, e.g., Equation (3.19) now becomes

$$0 = -81 - 144s^2 - 352s^4 - 256s^6 - 256s^8 + (144s^2 + 896s^4 + 256s^6)Y_H^2 - 256s^4Y_H^4.$$

We will have to treat Equations (3.16)-(3.21) accordingly. With this notation, our goal has become to calculate the characteristic polynomials for y_G , y_H and y_J . Elementary analytic geometry tells us that the connection between "old" coordinates (x_P, y_P) and "new" coordinates (X_P, Y_P) can be described by

$$(4.1) \quad \begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} x_F \\ y_F \end{pmatrix} + \frac{X_P}{2s} \begin{pmatrix} X \\ Y \end{pmatrix} + \frac{Y_P}{2s} \begin{pmatrix} -Y \\ X \end{pmatrix}$$

for $P \in \{G, H, J\}$, where $X = x_D - x_F$ and $Y = y_D - y_F$, as defined in Section 3.3. Thus,

$$(4.2) \quad 2s(y_P - y_F) = X_P \cdot Y + Y_P \cdot X.$$

Setting $z_P := y_P - y_F$ for the moment, successively using Equations (3.16)-(3.21), which described the connection between new coordinates of the points G, H, J , and the parameter s , and the equality $4s^2 = X^2 + Y^2$, after some calculations in the usual manner, we are able to deduce irreducible polynomials in $\mathbb{Z}[\sqrt{3}][z_P, X, Y]$ for $P \in \{G, H, J\}$. Using Equations (3.32) and (3.33) we are further able to eliminate X and Y , and deduce polynomials in $\mathbb{Z}[z_P, T]$ of total degree 188 for each of the points. Each of these splits again, leaving us with irreducible polynomials which are of degree 20 in T and total degree 20 for the points G, J and degree 24 in T and total degree 24 for the point H . Resubstituting $y_P - y_F$ for z_P and using (3.11) to eliminate y_F , we are left with integer polynomials in variables y_P and T of degree 112 for the points G, J and one of degree 128 for H , which this time split off irreducible polynomials of total degree 20 (for G and H) and 24 (for J), respectively. Calculating the resultant of these polynomials and P_T , thereby eliminating T , once more in each case we get polynomials of degree 176 for y_G , y_H , and y_J , respectively. Each contains among others an irreducible factor of degree 22 – the minimal polynomial, which we were looking for. Therefore we have:

Theorem 4.2. *The minimal polynomials for the y -coordinates of the vertices G, H , and J of the Harborth graph are*

$$P_{y_G} := -912811377667500 + 16117998953248125y_G^2 - 36709013218422600y_G^4 + 37940201286814800y_G^6 - 23463887481854208y_G^8 + 10021184125203456y_G^{10} - 3290335763447808y_G^{12} + 888521341648896y_G^{14} - 192809455583232y_G^{16} + 29839017902080y_G^{18} - 2742026240000y_G^{20} + 109337116672y_G^{22},$$

$$P_{y_H} := -12148787578527675 - 123412000423046805y_H^2 - 441020584930952232y_H^4 + 273168911377174014y_H^6 - 27343071784237320y_H^8 - 3667116898760364y_H^{10} + 823044986987616y_H^{12} - 32095868573376y_H^{14} - 4779985142784y_H^{16} + 615643279360y_H^{18} - 27098808320y_H^{20} + 427098112y_H^{22},$$

and

$$P_{y_J} := -9964518750000 + 570277711828125y_J^2 - 1780552966387500y_J^4 + 849106838377800y_J^6 + 644904447905880y_J^8 - 102048280254828y_J^{10} - 56106534718368y_J^{12} + 9027433758528y_J^{14} + 605520976896y_J^{16} - 103349145600y_J^{18} - 2815229952y_J^{20} + 427098112y_J^{22}.$$

5. MINIMAL POLYNOMIALS FOR THE x -COORDINATES

As we initially announced, we want to give minimal polynomials for all the coordinates of the most important vertices of the Harborth graph, where the center of origin is supposed to be the center of symmetry of the whole graph, i.e., the point K in Figure 2, and the axes are the axes of symmetry of the Harborth graph. This means that while the y -coordinates remain the same, when we shift our origin from A to K , the x -coordinates (with respect to the coordinate system centered in A) have to be shifted by $-x_J$, i.e., we have to set

$$(5.1) \quad x_P^{\text{new}} = x_P - x_J$$

for all points P , where the coordinates on the right hand side denote coordinates with respect to the origin A .

As a further difficulty, again we have to pay attention that up above the x -coordinates for the points G , H and J were given with respect to yet another, third coordinate system, which had F as its origin and was rotated when considered within the other two coordinate frames.

In the sequel, to avoid misunderstandings, we will switch notation, and denote all x -coordinates with respect to the system centered in A by x_P^{old} , those with respect to the system centered in K will now become x_P .

5.1. Coordinate transformations for the points A to F . Proceeding step by step as in Section 4.2, but starting from the second equation resulting from (4.1), i.e.,

$$(5.2) \quad 2s(x_J^{\text{old}} - x_F^{\text{old}}) = X_J \cdot X - Y_J \cdot Y,$$

we are able to deduce an irreducible polynomial $P_{x_J^{\text{old}}, T}$ in $\mathbb{Z}[\sqrt{3}][x_J^{\text{old}}, T]$ of total degree 24, and finally the characteristic polynomial of x_J^{old} of degree 22. Since it is an even polynomial⁶, it describes the coordinate $x_A = -x_J^{\text{old}}$ as well. Thus,

Theorem 5.1. *The minimal polynomial for the x -coordinate of the vertex A of the Harborth graph, where the coordinate system is the one given in Figure 1, is*

$$P_{x_A} := -830376562500 + 1358127000000x_A^2 - 34144387143750x_A^4 + 96857243056800x_A^6 - 68697978132015x_A^8 - 189712941147x_A^{10} + 6188723588664x_A^{12} - 704220643376x_A^{14} - 52577813248x_A^{16} + 27196394496x_A^{18} - 2918612992x_A^{20} + 106774528x_A^{22}.$$

With the help of the polynomial $P_{x_J^{\text{old}}, T}$ from above, we can produce a polynomial $P_{\text{trafo}} \in \mathbb{Z}[x_P, x_P^{\text{old}}, T]$ that describes the coordinate transformation (5.1), by calculating the resultant of $P_{x_J^{\text{old}}, T}$ and the polynomial $x_P - x_P^{\text{old}} + x_J^{\text{old}}$ with respect to x_J^{old} . This is of total degree 24, of degree 24 in T , and of degree 8 in both variables x_P and x_P^{old} .

Applying the same method as above, i.e., first calculating the resultant of this "transformation polynomial" and the respective polynomials $P_{x_P, T}$, which we now have to interpret as polynomials in the variables x_P^{old} and T , followed by a factorization over $\mathbb{Z}[\sqrt{3}]$, and finally repeating this process with the resulting polynomial and P_T , we are able to deduce the characteristic polynomials of the x -coordinates x_B to x_F . Since this procedure should be standard to the reader by now, we will

⁶We call a polynomial even if it only contains monomials of even degree.

not go into details anymore, but will only present the results. Once again we stress the fact that these polynomials are for x -coordinates with respect to K as origin:

Theorem 5.2. *The minimal polynomials for the x -coordinates of the vertices of the Harborth graph, where the coordinate system is the one given in Figure 1, are*

$$\begin{aligned}
P_{x_B} &:= -17372788157292129 + 85946816541669534x_B^2 - 172967171143553289x_B^4 + \\
&\quad 125428630440736260x_B^6 - 35361034276033728x_B^8 + 4402034757921792x_B^{10} - \\
&\quad 436015591392256x_B^{12} + 77220067192832x_B^{14} - 11054716223488x_B^{16} + \\
&\quad 874491412480x_B^{18} - 3473408000x_B^{20} + 557842432x_B^{22}, \\
P_{x_C} &:= -55268097000787592100 + 83653148035178006805x_C^2 - \\
&\quad 49933201015710366166x_C^4 + 15170804748275250138x_C^6 - \\
&\quad 2623723693990622868x_C^8 + 292733387369474292x_C^{10} - 24051159678783648x_C^{12} + \\
&\quad 1563610131071808x_C^{14} - 77064294460416x_C^{16} + 2572257472512x_C^{18} - \\
&\quad 50083921920x_C^{20} + 427098112x_C^{22}, \\
P_{x_D} &:= -15937557042969 + 69169635141939x_D^2 - 133600085051911x_D^4 + \\
&\quad 150590940104181x_D^6 - 109441808559384x_D^8 + 53597367271968x_D^{10} - \\
&\quad 17996039805696x_D^{12} + 4144963934208x_D^{14} - 647005151232x_D^{16} + \\
&\quad 66726690816x_D^{18} - 4293132288x_D^{20} + 139460608x_D^{22}, \\
P_{x_E} &:= -30534686672400 + 184473995962680x_E^2 - 493600710483009x_E^4 + \\
&\quad 800738068318020x_E^6 - 883225203916608x_E^8 + 687262746783744x_E^{10} - \\
&\quad 378024688788480x_E^{12} + 145061641105408x_E^{14} - 37695035736064x_E^{16} + \\
&\quad 6218402758656x_E^{18} - 582162055168x_E^{20} + 27334279168x_E^{22}, \\
P_{x_F} &:= -622521 + 20028276x_F^2 - 150285424x_F^4 - 349270464x_F^6 + 7694997504x_F^8 - \\
&\quad 5213620224x_F^{10} - 109200064512x_F^{12} + 709185896448x_F^{14} - 1112735219712x_F^{16} + \\
&\quad 387346071552x_F^{18} + 124822487040x_F^{20} + 8925478912x_F^{22}.
\end{aligned}$$

5.2. The "Coup de grâce" - the minimal polynomial for x_G . Due to the complexity of expressions we were not able to use the above procedure to calculate the characteristic polynomial for x_G - even with the help of Mathematica. Thus we have to resort to one final trick - yet another coordinate system. For this, first we mirror the Harborth configuration. After that we choose the point J as the new origin, and the ray JH as the positive part of the new x -axis (see Figure 4). Consequently, the new y -coordinates are the x -coordinates from above.

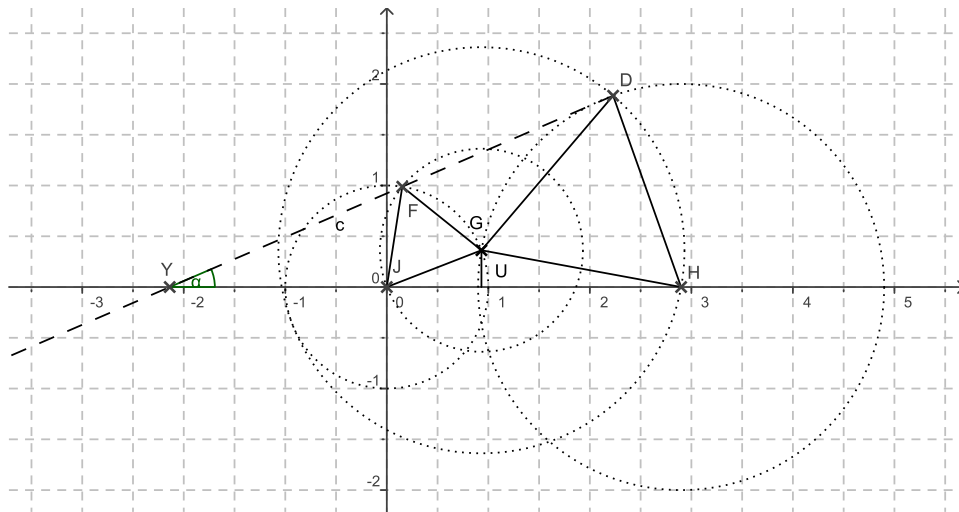
Let (u, U) be the coordinates of the point G with respect to this coordinate system. Using Equations (2.11) and (2.12), adapted to the new coordinate frame, we get

$$(5.3) \quad -3 + 4U^2 - 4Uy_F + 4y_F^2 = 0,$$

where y_F now denotes the y -coordinate of F with regard to this new system. Since y_F is equal to x_F of old, we thus have calculated a polynomial $P_{x_F, U}$. Calculating the resultant of this polynomial and P_{x_F} with respect to x_F leads to a polynomial in $\mathbb{Z}[U]$ of degree 44. Factoring this, we achieve an irreducible polynomial of degree 22, which is the minimal polynomial of U and thus of x_G .

Theorem 5.3. *The minimal polynomial of the x -coordinate of the vertex G of the Harborth graph is*

$$\begin{aligned}
P_{x_G} &:= -106929 + 9380331x_G^2 - 257190919x_G^4 + 2410771629x_G^6 - 11872837680x_G^8 + \\
&\quad 35430882432x_G^{10} - 66974055936x_G^{12} + 79549160448x_G^{14} - 56180293632x_G^{16} + \\
&\quad 20514865152x_G^{18} - 2573205504x_G^{20} + 139460608x_G^{22}.
\end{aligned}$$



Thus we have deduced minimal polynomials for all coordinates of the initial Harborth configuration, and consequently, because of the twofold symmetry and our particular, final choice of coordinates, for nearly all the vertices of the Harborth graph. This finishes our initial task.

With all the minimal polynomials at our disposal, we can finish this paper with some observations about their algebraic structure, and consequences for the Harborth graph.

Theorem 6.1. *For $P \in \{A, \dots, J\}$ let the coordinates $(\pm x_P, \pm y_P)$ of points in the Euclidean plane be particularly chosen roots of the irreducible polynomials P_{x_P}, P_{y_P} which were detailed in the previous sections. Then these coordinates satisfy the defining equations of coordinates of vertices of the Harborth graph.*

Second: Clearly Equations (2.5) - (2.18) have to be restated in accordance with our finally chosen coordinate system, which had the point K as its center. Thus, e.g., instead of (2.5) one has to show that the coordinates of the vertices A and B in the first quadrant satisfy $(x_B - x_A)^2 + y_B^2 = 1$, i.e., the points are at distance 1 from each other. Numeric approximations which determine our choice of roots have to be adapted as well.

Moreover calculations with the chosen coordinates are done in the sense of [9], whereat the computer algebra system has to take the full brunt of the work. That is: since there are no explicit formulas for the particular roots of the polynomials P_{x_P}, P_{y_P} , which we use as coordinates, we have to understand these roots as completely defined by their representing polynomials, and an isolating interval for each. Instead of an isolating interval one can equivalently use a numeric approximation of sufficient precision. In fact, this seems to be what the computer algebra system Mathematica does: it renders possible computations with algebraic numbers by way of `root` objects [15]. There, choosing a particular root actually means choosing one of the `root` objects produced by the `Solve` routine. This we did in accordance with the previously attained numerical results. The “proof” of the above theorem has been facilitated then by use of Mathematica’s inbuilt `RootReduce` routine, which had to be applied to the defining equations of the Harborth configuration. In order to achieve that calculations are finished within an acceptable time, the equations have to be expanded and written as sums of products. More concretely, in the example above, after having assigned particular `root` objects to the variables `xA`, `xB`, and `yB` with the help of the respective polynomials, one successively has to calculate `RootReduce[xA^2]`, `RootReduce[-2*xA*xB]`, etc. until one is able to put everything together, thus to be able to calculate `RootReduce[xA^2-2*xA*xB+xB^2+yB^2]`, which indeed results in 1. \square

Even though we have concentrated on one particular set of coordinates, our calculations have shown that, e.g., the y -coordinate y_B of the vertex B in any case must satisfy a polynomial equation. Thus there are only finitely many possibilities for coordinates of B . Since by the geometric construction, which we used, all other vertices can be shown to depend uniquely on this one initial vertex and its embedding in the Euclidean plane, different embeddings of the Harborth graph in the plane – if they were to exist at all – cannot be transformed into each other in a continuous way. In other words:

Theorem 6.2. *The Harborth graph is rigid.*

Finally, closer scrutiny of the minimal polynomials leads to

Lemma 6.3. *Let z_P be one of the coordinates of a vertex P of the Harborth graph different from zero. Then its minimal polynomial $P_{z_P} \in \mathbb{Z}[X]$ is an even polynomial of degree 22 and signature (6, 8). That is, it has 6 real zeros and 8 distinct pairs of conjugate complex zeros. Consequently we have,*

$$(6.1) \quad P_{z_P} = F_{z_P} \circ G$$

where $G = X^2 \in \mathbb{Z}[X]$, \circ denotes composition and F_{z_P} is a uniquely determined irreducible integer polynomial of degree 11 and signature (3, 4).

A precise proof for this lemma can be done by using any one of the existing algorithms for root isolation and some basic calculus.

Theorem 6.4. *The coordinates of vertices of the Harborth graph which are different from zero cannot be expressed in terms of radicals. Furthermore the Harborth graph as a whole cannot be constructed by compass and ruler alone.*

Proof. A well known corollary in Galois theory⁷ tells us, that the zeros of a polynomial of odd prime degree, which is irreducible over a real number field, are expressible in terms of radicals, if and only if either the polynomial has only one real zero, or all of its zeros are real. We have already ascertained that the polynomials F_{z_P} appearing in Equation (6.1) each have 3 distinct real roots. Thus the equations $F_{z_P} = 0$, and consequently $P_{z_P} = 0$ are not soluble by radicals. This proves the first assumption.

We show the second assumption by an indirect proof: suppose that the Harborth graph were constructible by compass and ruler. This would imply that the vertex A , thus its x -coordinate x_A and furthermore x_A^2 would be constructible by compass and ruler. As a consequence of this, the order of the Galois group of the corresponding minimal polynomial, which is F_{x_A} , would be a power of 2, and the Galois group would be soluble⁸. This, as we have just seen above, is absurd, completing the proof of the theorem. \square

REFERENCES

1. Emil Artin, *Galoissche Theorie*, Verlag Harri Deutsch, Zürich, 1968.
2. Henri Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics, vol. 138, Springer Verlag, Berlin, 1993.
3. David Cox, John Little, and Donal O'Shea, *Ideals, Varieties, and Algorithms*, 2 ed., Springer Verlag, New York, 1997.
4. Heiko Harborth, *Problem 20 : Ein Streichholzproblem*, Math. Semesterber. **32** (1985), 276–277.
5. Heiko Harborth, *Match Sticks in the Plane*, The Lighter Side of Mathematics. Proceedings of the Eugène Strens Memorial Conference of Recreational Mathematics & its History, Calgary, Canada, July 27 - August 2, 1986 (Washington) (Richard K. Guy and Robert E. Woodrow, eds.), Spectrum Series, The Mathematical Association of America, 1994, pp. 281–288.
6. Heiko Harborth and Arnfried Kemnitz, *Integral Representations of Graphs*, Contemporary Methods in Graph Theory (Mannheim) (Rainer Bodendiek, ed.), B.I.-Wissenschaftsverlag, 1990, pp. 359–367.
7. Nora Hartsfield and Gerhard Ringel, *Pearls in Graph Theory - A Comprehensive Introduction*, 1 ed., Academic Press, Boston, 1990.
8. Markus Hohenwarter and Yves Kreis, *GeoGebra*, <http://www.geogebra.at>, 2006, viewed October 3, 2006.
9. Rüdiger Loos, *Computing in Algebraic Extensions*, Computer Algebra. Symbolic and Algebraic Computation (B. Buchberger, G.E. Collins, R. Loos, and R. Albrecht, eds.), Springer-Verlag, Wien, 2 ed., 1983, pp. 173–187.
10. Harald Löwe, *Private communication*, February 2003.
11. Ed Pegg, *mathpuzzle - Material added 8 Jan 06 (Happy New Year)*, <http://www.mathpuzzle.com/26Feb2006.html>, January 2006, viewed October 3, 2006.
12. Ivar Peterson, *Ivar Peterson's Mathland - Matchsticks in the Summer*, http://www.sciencenews.org/pages/sn_arch/8_10_96/mathland.htm, August 1996, viewed October 3, 2006.
13. András Recski, *Combinatorics in Electrical Engineering and Statics*, Handbook of Combinatorics (R. Graham, M. Grötschel, and L. Lovász, eds.), Elsevier Science, 1995, pp. 1911–1924.
14. Saltire Software, *Harborth Graph - Request from Joint Math Meetings*, <http://www.geometryexpressions.com/forum/showthread.php?tid=5>, May 2006, viewed October 3, 2006.
15. Stephen Wolfram, *Mathematica 5.2 Documentation. 3.4.3 Advanced Topic: Algebraic Numbers*, <http://documents.wolfram.com/mathematica/book/section-3.4.3>, 2006, viewed September 25, 2006.

⁷See the corollary to Theorem 46 in [1].

⁸See Theorem 47 in [1].

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